THE ASYMPTOTIC RELIABILITY OF PARALLEL (SERIES) AND SERIES (PARALLEL) SYSTEMS

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THESIS

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ABSTRACT

Parallel(Series) and Series(Parallel) systems are defined, and a physical basis is presented as motivation for analysis of the asymptotic properties of their reliabilities. Two theorems are presented to provide useful computational tools in evaluating reliability limit functions. The behavior of balanced and unbalanced systems is examined and conclusions drawn therefrom. A special imbalance which yields a unique limit function is presented.

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I. INTRODUCTION

The study of system reliability may be approached in various ways. As technology progresses, so does the sophistication of man-machine systems. The ever increasing complexity of machines and equipment makes the task of computing system reliability more difficult.

Certain types of systems lend themselves to relatively easy calculation of exact reliabilities. Similarly certain theoretical or idealized systems may possess rather elementary exact reliability expressions. In some cases theoretical systems may be used to approximate physical systems. The approximations afforded by theoretical models are usually sufficient during many phases of systems development when order of magnitude accuracy is all that is required by an analyst or engineer.

Consider a simple system which is made up of a number of components, say L, connected in series. One might reasonably desire to increase the reliability of this series path by redundant construction. Two or more, say W, identical series paths connected in parallel would increase the reliability of the path. For the redundant system to operate it is necessary that only one series path function. An idealized version of such a system of series paths in parallel is called Parallel(Series). Figure 1 is a schematic representation of a Parallel(Series), or P(S), system. For

simplicity in Figure 1 and throughout this paper, component reliabilities, p, are assumed to be equal for all components in a system. In addition, independence of components is assumed. The exact reliability of a Parallel(Series) system is

$$h(p,L,W) = 1 - (1-p^L)^W. \tag{1-1}$$

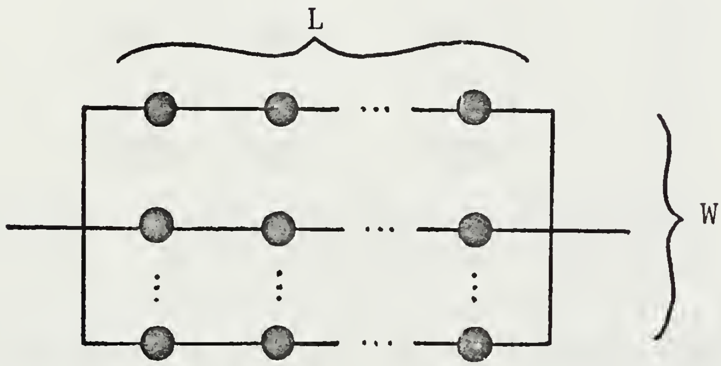

Figure 1. Schematic Representation of a Parallel(Series) System

The Parallel(Series) system is an example of redundancy at the "system level." Another type of redundancy can be introduced at the "component level." In this case identical components are placed in parallel with each other inside the series path. Practically, the number of paralleled components at any stage of the system would depend on the component reliability, that is, the degree of redundancy required to achieve a desired level of stage reliability. For the system to operate it is necessary that at least one component in each stage function. An idealization of such a system is called a Series(Parallel) system. Figure 2 is a schematic representation of a Series(Parallel), or S(P),

system. The exact reliability of a Series(Parallel) System is

$$h(p,L,W) = (1-(1-p)^W)^L. \tag{1-2}$$

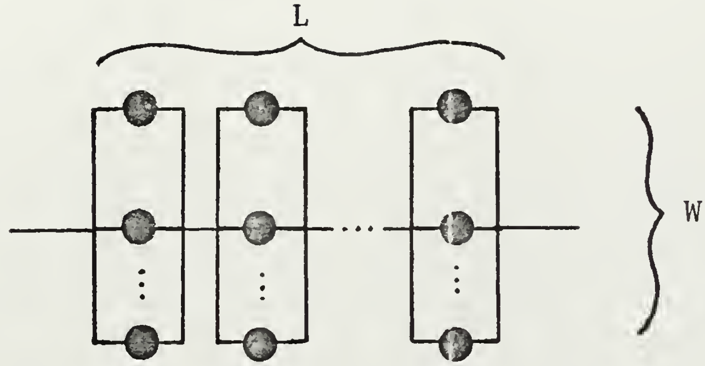

Figure 2. Schematic Representation of a Series(Parallel) System

In the preceding discussion of Parallel(Series) and Series(Parallel) systems, the dimension L is the length in terms of the number of components which constitute a series path. The dimension W is the width of the system, or the number of redundant paths in the case of Parallel(Series) systems, and the number of redundant components in each stage of a Series(Parallel) system.

A well known duality exists between the Parallel(Series) and Series(Parallel) classes of systems. For simplicity in this paper, Series(Parallel) systems will be dealt with explicitly when necessary rather than relying on duality arguments.

In succeeding sections of this paper the asymptotic, or limiting behavior of Parallel(Series) and Series(Parallel) systems is examined. In Section II two theorems for computation of reliability limit functions are proved and applied

to balanced systems, i.e., those for which L=W. Section III is devoted to systems where an imbalance has been introduced. In Section IV a special imbalance which yielded a unique limit function is covered.

II. BALANCED SYSTEMS

For the purposes of this paper, it is convenient to consider the dimensions of a system as increasing functions of some auxillary variable, n. Length and width will be expressed as L(n) and W(n), respectively. Thus, for a given component reliability $0 \leq p \leq 1$, system reliability is a function of one variable, n. The exact reliability equations (1-1) and (1-2) become respectively,

$$h(p,n) = 1-(1-p^{L(n)})^{W(n)}, \tag{2-1}$$

and

$$h(p,n) = \{1-(1-p)^{W(n)}\}^{L(n)}. \tag{2-2}$$

For both equations (2-1) and (2-2), $h(0,n)=0$ and $h(1,n)=1$. Succeeding discussions will consider values of component reliability, $0 < p < 1$.

A balanced system is one in which the width, or degree of redundancy, is equal to the length. For balanced systems considered in this paper, let $L(n)=W(n)=n$.

A. PARALLEL(SERIES) SYSTEMS

Equations (2-1) and (2-2) describe sequences of functions for successive values of n. These sequences approach limits as n becomes infinitely large. Define the limit of either of these sequences as follows:

$$H(p) = \lim_{n\to\infty} h(p,n) . \tag{2-3}$$

The following definitions will be useful in succeeding developments:

$$f(p,n) = W(n)p^{L(n)}, \qquad (2\text{-}4)$$

and

$$f_o(p) = \lim_{n\to\infty} f(p,n) = \lim_{n\to\infty} W(n)p^{L(n)}. \qquad (2\text{-}5)$$

The following theorem is the basic tool for determining the limit function for a sequence of P(S) systems:

<u>THEOREM 2.1</u>. For a Parallel(Series) system, if both $L(n)$ and $W(n)\to\infty$ as $n\to\infty$, then the system limit function is given by

$$H(p) = 1 - e^{-f_o(p)}. \qquad (2\text{-}6)$$

Proof:

From equation (2-1) the reliability of a Parallel-(Series) system is

$$h(p,n) = 1-(1-p^{L(n)})^{W(n)}. \qquad (2\text{-}7)$$

Rearranging

$$1-h(p,n) = (1-p^{L(n)})^{W(n)},$$

and

$$\log_e(1-h(p,n)) = W(n)\log_e(1-p^{L(n)})$$

$$= W(n)p^{L(n)} \frac{\log_e(1-p^{L(n)})}{p^{L(n)}}.$$

Taking limits

$$\lim_{n\to\infty} \log_e(1-h(p,n)) = -\lim_{n\to\infty} W(n)p^{L(n)} = -\lim_{n\to\infty} f(p,n), \qquad (2\text{-}8)$$

since as $L(n)\to\infty$, $p^{L(n)}\to 0$ and $\dfrac{\log_e(1-p^{L(n)})}{p^{L(n)}} \to -1$.

The right hand side of equation (2-8) is $-f_o(p)$ from equation (2-5). Taking anti-logarithms equation (2-8) becomes

$$\lim_{n\to\infty} (1-h(p,n)) = \exp(-f_o(p)), \tag{2-9}$$

or

$$1-\lim_{n\to\infty} h(p,n) = 1-H(p) = \exp(-f_o(p)). \tag{2-10}$$

Thus the limit function is

$$H(p) = 1-e^{-f_o(p)}, \qquad \text{QED.} \tag{2-11}$$

Theorem 2.1 provides a simple computational method for determining limit functions for Parallel(Series) systems. The utility of this theorem is demonstrated for a balanced system.

A balanced system was defined as one for which $L(n) = W(n) = n$. The reliability of the system is then

$$h(p,L,W) = h(p,n) = 1-(1-p^n)^n.$$

Applying Theorem 2.1

$$f(p,n) = np^n,$$

and

$$f_o(p) = \lim_{n\to\infty} np^n = 0.$$

Thus

$$H(p) = 1-e^{-0} = 0, \quad 0 < p < 1. \tag{2-12}$$

Combining equation (2-12) with the values for H(0) and H(1), the limit function for a balanced Parallel(Series) system is

$$H(p) \begin{array}{ll} = 0\ , & 0 \leq p < 1 \\ = 1\ , & p = 1\ . \end{array} \tag{2-13}$$

Thus for any component reliability $p<1$, a balanced Parallel-(Series) system becomes perfectly unreliable as the size of the system becomes infinitely large. Figure 3 illustrates the behavior of balanced Parallel(Series) systems for selected values of n. It is apparent from Figure 3 that the rate at which the sequence of functions approaches the limit function is non-uniform.

B. SERIES(PARALLEL) SYSTEMS

The asymptotic behavior of Series(Parallel) systems follows by an argument analogous so that for P(S) systems. First, define the following functions:

$$g(p,n) = L(n)(1-p)^{W(n)} , \qquad (2\text{-}14)$$

and

$$g_o(p) = \lim_{n\to\infty} g(p,n)$$

$$= \lim_{n\to\infty} L(n)(1-p)^{W(n)} . \qquad (2\text{-}15)$$

The analogue to Theorem 2.1 is

<u>THEOREM 2.2.</u> For a Series(Parallel) system, if both $L(n)$ and $W(n)\to\infty$ as $n\to\infty$, then the system limit function is given by

$$H(p) = e^{-g_o(p)} . \qquad (2\text{-}16)$$

Proof:

From equation (2-2) the reliability of a S(P) system is

$$h(p,n) = (1-(1-p)^{W(n)})^{L(n)} . \qquad (2\text{-}17)$$

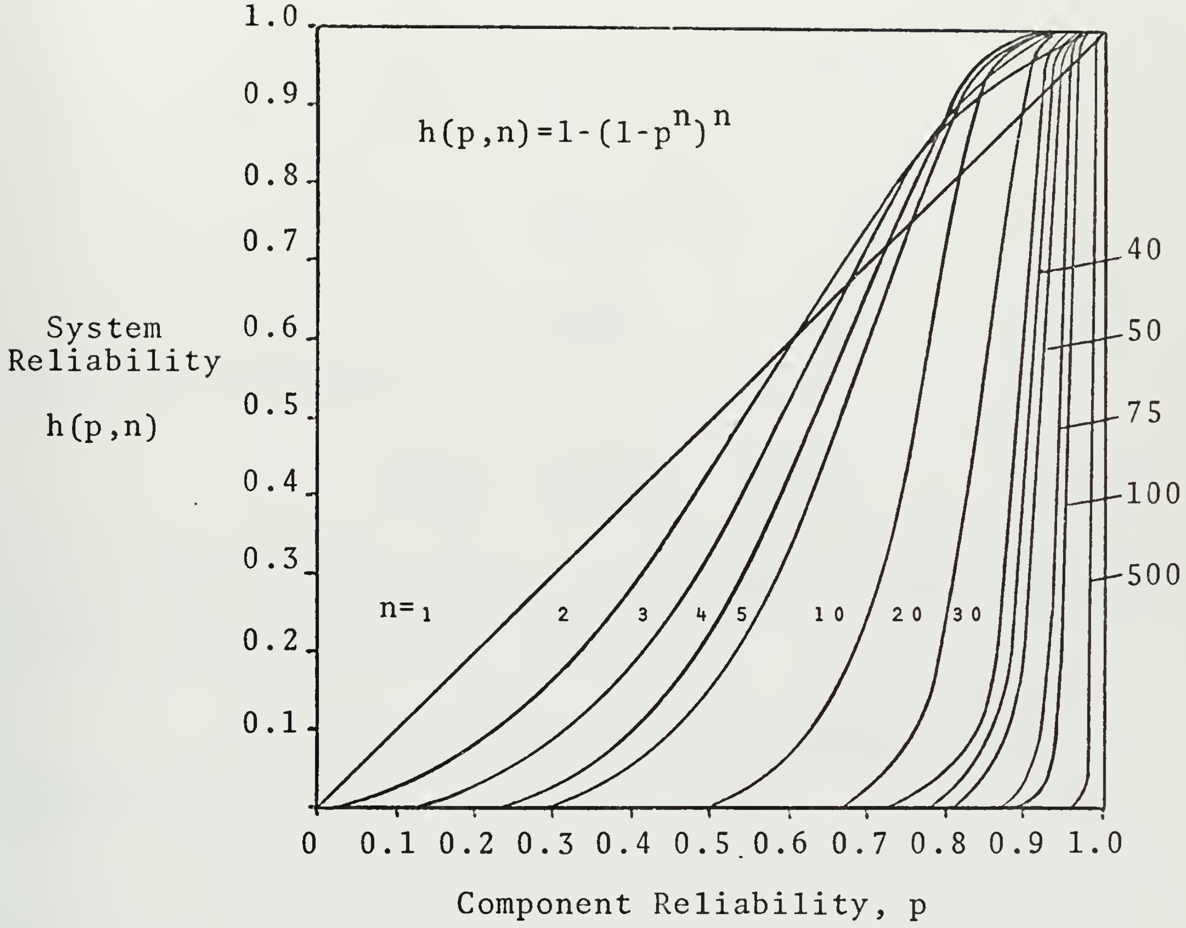

Figure 3. Reliability of P(S) Systems for Selected Values of n

Taking logarithms

$$\log_e h(p,n) = L(n)\log_e(1-(1-p)^{W(n)})$$

$$= L(n)(1-p)^{W(n)} \frac{\log_e(1-(1-p)^{W(n)})}{(1-p)^{W(n)}} .$$

Taking limits

$$\lim_{n\to\infty} \log_e h(p,n) = -\lim_{n\to\infty} L(n)(1-p)^{W(n)}$$

$$= -\lim_{n\to\infty} g(p,n)$$

$$= -g_o(p) . \tag{2-18}$$

Thus the limit function is

$$H(p) = e^{-g_o(p)}, \quad \text{QED.} \tag{2-19}$$

The utility of Theorem 2.2 is demonstrated for the balanced system. Let $L(n)=W(n)=n$. Then

$$h(p,L,W) = h(p,n) = (1-(1-p)^n)^n.$$

Applying Theorem 2.2

$$g(p,n) = n(1-p)^n ,$$

and

$$g_o = 0.$$

Thus

$$H(p) = e^{-0} = 1, \quad 0 < p < 1. \tag{2-20}$$

Combining equation (2-20) with values for H(0) and H(1), the limit function for a balanced Series(Parallel) system is

$$H(p) \begin{array}{ll} = 0 , & p=0 \\ = 1 , & 0 < p \leq 1. \end{array} \tag{2-21}$$

Thus for any component reliability $p>0$, a balanced S(P) system becomes perfectly reliable as the size of the system becomes infinitely large. Figure 4 illustrates the behavior of the balanced S(P) system for selected values of n. As with the P(S) system, the rate of convergence to the limit function is non-uniform.

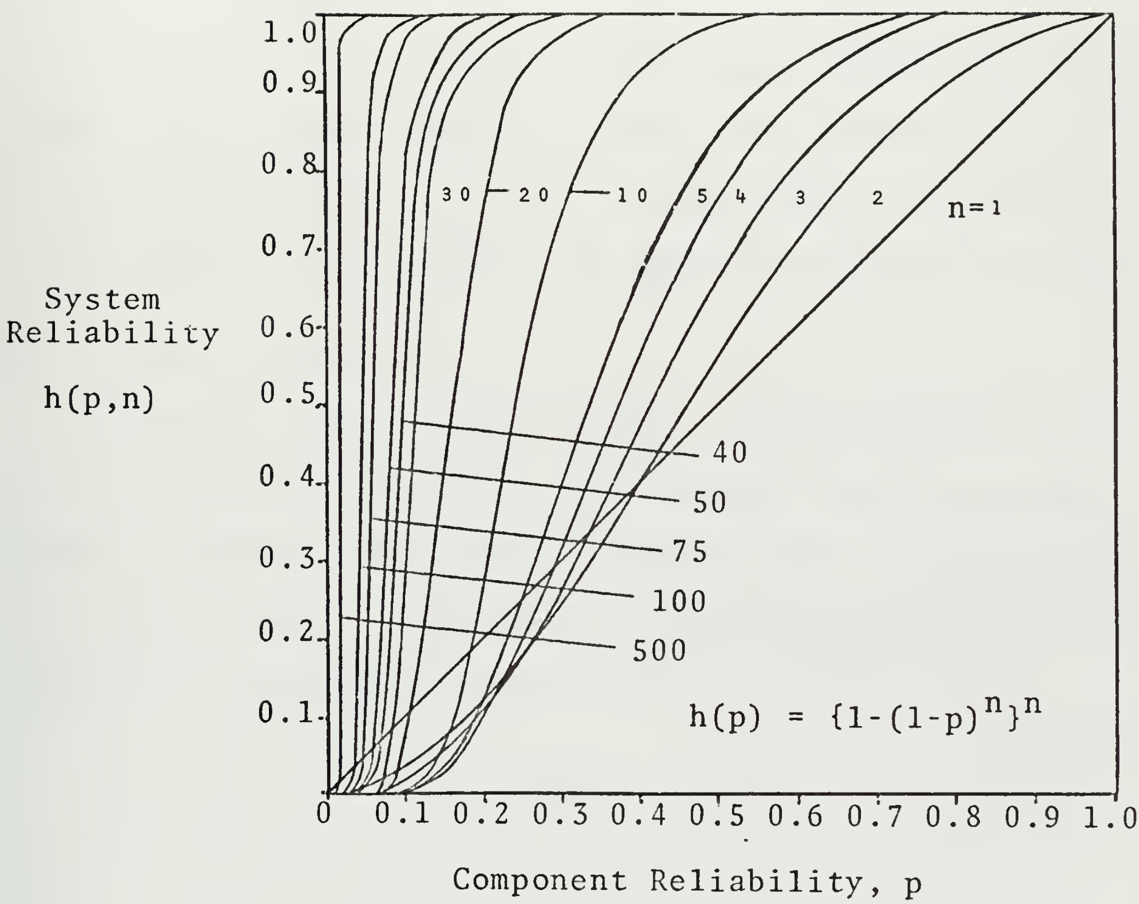

Figure 4. Reliability of S(P) Systems for Selected Values of n

III. UNBALANCED SYSTEMS

The remaining discussion considers only the Parallel-(Series) systems explicitly. Corresponding results for Series(Parallel) systems may be readily obtained through analogous arguments.

Consider now systems in which length and width are not equal, that is unbalanced systems. The question to be answered is, "Is there some ratio of imbalance for which the limit function will differ from that for balanced systems?"

A. LENGTH DOMINANCE

Consider now two P(S) systems. Suppose System 1 is balanced and $L_1(n)=W_1(n)=n$. Let System 2 have the same width as System 1 and greater length, that is $W_2(n)=W_1(n)=n$ and $L_2(n)>L_1(n)=n$. Comparison of the two systems was made through the following inequalities:

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\begin{aligned}
n &< L_2(n) \\
p^n &> p^{L_2(n)} \\
1-p^n &< 1-p^{L_2(n)} \\
(1-p^n)^n &< (1-p^{L_2(n)})^n \\
1-(1-p^n)^n &> 1-(1-p^{L_2(n)})^n \qquad (3\text{-}1)
\end{aligned}
$$

or

$$h_1(p,n) > h_2(p,n).$$

Inequality (3-1) shows that any length dominated P(S) system will have a reliability less than the corresponding balanced system. Thus, the balanced P(S) system may be viewed as an

upper bound to the length dominated P(S) system. The limit function for System 2 and length dominated systems in general is

why?

$$\lim_{n\to\infty} h_2(p,n) \begin{cases} = 0, & 0 \le p < 1 \\ = 1, & p = 1. \end{cases} \qquad (3\text{-}2)$$

B. WIDTH DOMINANCE

Suppose now that in System 2 described above, width exceeds length. Let $L_2(n)=L_1(n)=n$ and $W_2(n)>W_1(n)=n$. Consider the following inequalities:

$$\begin{aligned} n &< W_2(n) \\ (1-p^n)^n &> (1-p^n)^{W_2(n)} \\ 1-(1-p^n)^n &< 1-(1-p^n)^{W_2(n)} \end{aligned} \qquad (3\text{-}3)$$

or

$$h_1(p,n) < h_2(p,n).$$

Inequality (3-3) shows that width dominance enhances the reliability of P(S) systems. The balanced P(S) system may be viewed as a lower bound for the corresponding width dominated system.

Consider an unbalanced system with dimension ratio $L(n):W(n)=n:kn$. Applying Theorem 2.1

l

$$f(p,n)=(kn)p^n, \qquad (k>1, \text{ positive integer})$$

and

w

$$f_o(p)=0.$$

The limit function for this sequence is then

$$H(p) \begin{cases} = 0, & 0 \le p < 1 \\ = 1, & p = 1, \end{cases}$$

which is identical to the limit function shown in equation (2-13) for balanced systems. Similar results can be obtained for dimension ratios such as $L(n):W(n)=n:n^k$, where $k>1$ is a positive integer.

Applying Theorem 2.1 to width dominated systems yields insight as to how limit functions which are not zero for all $p<1$ can be achieved.

Recall definitions (2-4) and (2-5) preceding Theorem 2.1:

$$f_o(p) = \lim_{n\to\infty} f(p,n) = \lim_{n\to\infty} W(n)p^{L(n)}. \qquad (3\text{-}4)$$

It is apparent that in order to achieve a sequence which does not converge to zero for all $p<1$, the value of width relative to length must be such that $W(n)$ overdrives $L(n)$ and it's effect on p.

As previously shown for balanced and length dominated systems, $f(p,n)$ converges to zero for all $p<1$. It is desirable to find some ratio of $W(n)$ to $L(n)$ such that

$$0<f_o(p)<\infty \qquad (3\text{-}5)$$

holds, for if such values of $f_o(p)$ exist, then the limit function will converge to some non-zero value.

IV. SYSTEMS WITH A SPECIAL IMBALANCE

In Section III it was stated that if any P(S) system configuration were to yield a limit function different from

$$H(p) \begin{cases} = 0, & 0 \leq p < 1 \\ = 1, & p = 1, \end{cases} \tag{4-1}$$

then width must dominate to the extent that it overdrives the effect of length.

For a dimension ratio

$$L(n):W(n)=n:k^n \tag{4-2}$$

the desired effect can be achieved. Equation (3-3) becomes

$$f_o(p) = \lim_{n\to\infty} W(n)p^{L(n)} = \lim_{n\to\infty}(k^n)p^n = \lim_{n\to\infty} (kp)^n, \tag{4-3}$$

and

$$f_o(p) \begin{cases} = 0, & p < 1/k \\ = 1, & p = 1/k \\ = \infty, & p > 1/k. \end{cases} \tag{4-4}$$

Applying Theorem 2.1, the limit function is

$$H(p) \begin{cases} = 0, & 0 \leq p < 1/k \\ = 1-e^{-1}, & p = 1/k \\ = 1, & 1/k < p \leq 1. \end{cases} \tag{4-5}$$

For selected values of $k>1$ a family of limit functions $H(p)$ is shown in Figure 5.

Figure 5 illustrates an interesting result: extremely poor quality components may be grouped so as to create a highly reliable system. While the idea of using redundancy to compensate for low reliability components is not new, the

manner of combination and relative proportions of width versus length may be useful, especially as the technology of micro-miniaturization progresses.

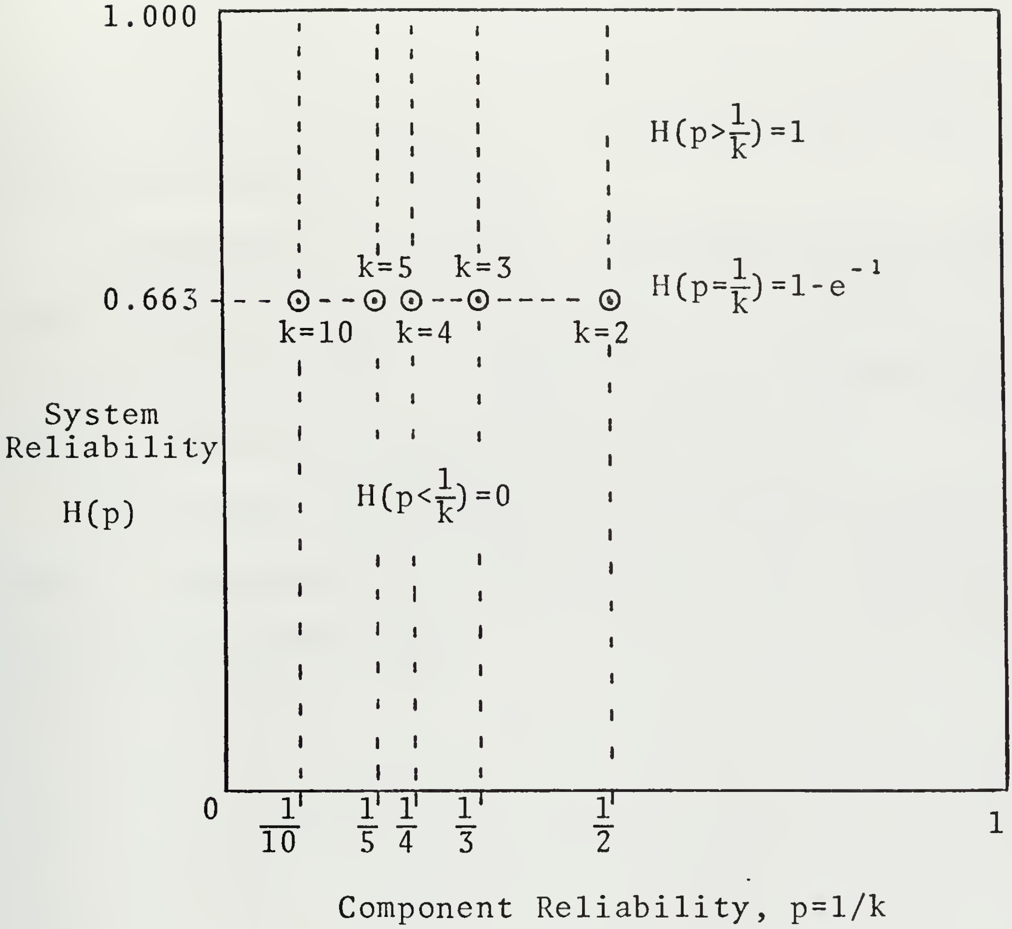

Figure 5. Limit Function for Special Type of Imbalance in Width Dominance

V. CONCLUSION

A physical basis for Parallel(Series) and Series-(Parallel) systems was presented as motivation for analysis of the asymptotic properties of these systems.

Two theorems were presented to provide useful computational tools for calculating the reliability limit functions for these classes of systems.

The behavior of balanced and unbalanced systems was investigated and general implications of balance versus imbalance were stated. A special type of imbalance was found which yielded an interesting limit function.

The latter case suggests the possibility that other special imbalances will also produce limit functions which are equally interesting.

BIBLIOGRAPHY

1. Barlow, R. E. and Proschan, F., Mathematical Theory of Reliability, p. 186-190, Wiley, 1965.

2. Esary, J. D., Classroom Notes, Theory of Minimal Cut-Lower Bound Reliability Approximations, presented to students at the Naval Postgraduate School, Monterey, California, November 1970.

3. Kaplan, W., Advanced Calculus, p. 305-309, 337-342, Addison-Wesley, 1952.

4. Rau, J. G., Optimization and Probability in Systems Engineering, p. 167-172, Van Nostrand Reinhold, 1970.

5. Taylor, A. E., Advanced Calculus, p. 586-595, Ginn, 1955.

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